Memo on computing Stokes coefficients of the expansion of the atmosphere contribution to the geopotential into a series of spherical harmonics

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1 Definitions

Let's define spherical harmonic of degree n, order m this way:

$$Y_n^m(\varphi, \lambda) = R_n^m \, P_n^m(\varphi) \, e^{im\lambda} \tag{1}$$

where φ is the geocentric latitude in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$, positive towards north, λ — longitude positive towards east, and $P_n^m(\varphi)$ is the associated Legendre polynomial:

$$P_n^m(\varphi) = \frac{(-1)^{n+m}}{2^n n!} (\sin\varphi)^m \frac{d^{n+m}}{d(\sin\varphi)^{n+m}} (\sin^2\varphi - 1)^n \tag{2}$$

and R_n^m is a normalization factor

$$\begin{aligned} \text{for } m = 0 \quad R_n^m &= \sqrt{2n+1} \\ \text{for } m > 0 \quad R_n^m &= \sqrt{2(2n+1)\,\frac{(n-m)!}{(n+m)!}} \end{aligned} \tag{3}$$

The spherical harmonics defined this way satisfy the following normalization:

$$\iint\limits_{\Omega} \mathrm{Re} Y_n^m(\varphi, \lambda) \, \mathrm{Re} Y_{n'}^{m'}(\varphi, \lambda) \, d\varphi \, d\lambda = 4\pi \, \delta(n', n) \, \delta(m', m) \tag{4}$$

$$\iint\limits_{\Omega} \mathrm{Im} Y_n^m(\varphi, \lambda) \, \mathrm{Im} Y_{n'}^{m'}(\varphi, \lambda) \, d\varphi \, d\lambda = 4\pi \, \delta(n', n) \, \delta(m', m) \tag{5}$$

Here integration is done over the entire sphere, and δ is the Kronecker symbol. The spherical harmonics normalized this clumsy way are called "4π fully normalized".

2 Stokes coefficients

For a thin-layer atmosphere acting on the spherical Earth surface ($r = R_\oplus$), the complex Stokes coefficients of the expansion of the atmosphere contribution to the geopotential into a series of spherical harmonics are equal to:

$$A_n^m(t) = \frac{3}{4\pi \bar{R} \rho_\oplus g_o} \frac{1+k'_n}{2n+1} \iint\limits_\Omega P(t,\varphi,\lambda)\, Y_n^m(\varphi,\lambda)\, \cos\varphi \, d\varphi \, d\lambda \tag{6}$$

where $\bar{R}\rho_\oplus$andg_o are respectively the mean Earth radius (6371 km), density (5515 kg/m^3) and surface gravity (9.80665 m/s^2), $P(t,\varphi,\lambda)$ is the surface pressure field, k'_n is the load Love number of the n th degree.

The Love numbers are computed in the reference frame of the the total Earth system: solid Earth and atmosphere.

The oceanic response to atmospheric pressure forcing as an inverted barometer (IB):

$$\Delta P_a + \Delta P_w - \Delta \bar{P}_o = 0 \tag{7}$$

where ΔP_a is the variation of local atmosphere pressure, ΔP_w is the local variation of the ocean bottom pressure due to induced sea level change, and $\Delta \bar{P}_o$ is the mean atmosphere pressure over the world's oceans:

$$\Delta \bar{P}_o = \frac{\iint\limits_{ocean} \Delta P(\vec{r}', t) \cos\varphi' d\lambda' d\varphi'}{\iint\limits_{ocean} \cos\varphi' d\lambda' d\varphi'} \tag{8}$$

which is applied uniformly at the sea floor [*van Dam and Wahr*(1987)]. This term is introduced in equation 7 in order to enforce conservation of ocean mass. Thus, the total ocean bottom pressure, $\Delta P_a + \Delta P_w$, is described by equation 8. It has been shown in numerous studies (see, for example, [*Tierney et al.*(2000)]) that this model adequately describes the sea height variations for periods longer than 5–20 days. However, the ocean response significantly deviates from the IB hypothesis for shorter periods [*Wunsch and Stammer*(1997)].

Since $\Delta \bar{P}_o$ is zero over the land and depends only on time over the world's oceans, it is convenient to split integral 6 into a sum of integrals over the ocean and over the continental surface. In our computation we use the land-sea mask from the FES99 [*Lefèvre et al.*(2002)] ocean tidal model with a 0°.25 spatial resolution.

Since the NCEP Reanalysis numerical weather models have a time resolution of 6 hours, the semidiurnal (S_2) atmospheric tide induced by solar heating cannot be modeled correctly, because its frequency corresponds exactly to the Nyquist frequency. The diurnal (S_1) atmospheric tide is somewhat distorted as well, because of the presence of the ter-diurnal signal, which is folded into the diurnal frequency due to sampling. We compute this signal over the period [1980.0, 2002.0] for each cell at the gird and subtract it from the pressure. Thus, our time series has zero mean and no signal at S_1 and S_2 frequencies. Contribution due to atmospheric tides **must be added** to our time series.

3 Recurrent relationships for computation of spherical harmonics

Computation of the spherical harmonics $Y_n^m(\varphi)$ is performed using the following recurrent relationships:

- m=0, n=0

  $P_0^0(\varphi) = 1$

- m=0, n=1

  $P_1^0(\varphi) = \sin\varphi$

- m=n, $n > 1$

  $P_n^m(\varphi) = (2m-1)\,\cos\varphi\,P_{n-1}^{m-1}(\varphi)$

- $\forall m, n = m+1$

  $P_n^m(\varphi) = (2m+1)\,\sin\varphi\,P_{n-1}^{m}(\varphi)$

- $\forall m, n > m+1$

  $$P_n^m(\varphi) = \frac{2n-1}{n-m}\,P_{n-1}^m(\varphi) \;-\; \frac{n+m-1}{n-m}\,P_{n-2}^m(\varphi)$$

- $m = 0, \forall n$

  $$R_n^0 = \sqrt{\frac{2n+1}{4\pi}}$$

- $m = n, n > 0$

  $$R_m^n = \frac{\sqrt{1+\dfrac{1}{2n}}}{2n-1}\,R_{m-1}^{n-1}$$

- $\forall m, n > 1$

  $$R_n^m = \sqrt{\frac{(2n+1)(n-m)}{(2n-1)(n+m)}}\,R_{n-1}^m$$

- $m = 0, \forall n$

  $C_0 = 1$

  $S_0 = 0$

- $m = 1, \forall n$

  $C_1 = \cos\lambda$

  $S_1 = \sin\lambda$

- $m > 1, \forall m$

  $C_m = 2\cos\lambda\,C_{m-1} - C_{m-2}$

  $S_m = 2\cos\lambda\,S_{m-1} - S_{m-2}$

- $\forall m, \forall n$

$$\operatorname{Re} Y_n^m = R_n^m \, P_n^m \, C_m$$
$$\operatorname{Im} Y_n^m = R_n^m \, P_n^m \, S_m$$

4 Computation of spherical harmonics

First, we compute normalization coefficients:

for m = 0, ... M
    for n = m, ... M
    if n = m then
        if n = 0 then $R_n^m = 1$
        else if n = 1 then $R_n^m = \dfrac{\sqrt{2 + \dfrac{1}{n}}}{2n-1} R_{n-1}^{m-1}$
        else $R_n^m = \dfrac{\sqrt{1 + \dfrac{1}{2n}}}{2n-1} R_{n-1}^{m-1}$
    else
        $R_n^m = \sqrt{\dfrac{(2n+1)(n-m)}{(2n-1)(n+m)}} R_{n-1}^m$
    end if
    end do
end do

Then for each center of the cell φ, λ on the grid which is land we compute the spherical harmonics. Computation of the spherical harmonics matrix is performed by raws (orders):

$$\begin{array}{ccccccc} P_0^0 & P_1^0 & P_2^0 & P_3^0 & P_4^0 & & \text{order } 0 \\ & P_1^1 & P_2^1 & P_3^1 & P_4^1 & & \text{order } 1 \\ & & P_2^2 & P_3^2 & P_4^2 & & \text{order } 2 \\ & & & P_3^3 & P_4^3 & & \text{order } 3 \\ & & & & P_4^4 & & \text{order } 4 \end{array} \tag{9}$$

Within each raw the diagonal and above-diagonal element is expressed through the diagonal element of the previous raw (order). Other elements of the same order are expressed through two previous elements.

$C_0 = 1$
$S_0 = 0$
$C_1 = \cos\lambda$
$S_1 = \sin\lambda$
$P_0^0 = 1$
$P_0^1 = \sin\varphi$
$P_1^1 = \cos\varphi$
for k = 2, ... M
    $C_k = 2\cos\lambda \, C_{k-1} - C_{k-2}$

$S_k = 2\cos\lambda\, S_{k-1} - S_{k-2}$
$P_k^k = (2k-1)\,\cos\varphi\, P_{k-1}^{k-1}$
$P_k^{k-1} = (2k-1)\,\sin\varphi\, P_{k-1}^{k-1}$
end do

for m = 0, ... M
$\alpha = 0$
$\beta = 2m - 1$

$\mathrm{Re}\, Y_m^m = R_n^m\, P_m^m\, C_m$
$\mathrm{Im}\, Y_m^m = R_n^m\, P_m^m\, S_m$
$\alpha = \alpha + 1$
$\beta = \beta + 1$

if $m < M$
$\mathrm{Re}\, Y_{m+1}^m = R_n^{m+1}\, P_{m+1}^m\, C_m$
$\mathrm{Im}\, Y_{m+1}^m = R_n^{m+1}\, P_{m+1}^m\, S_m$
$\alpha = \alpha + 1$
$\beta = \beta + 1$
endif

for n = m, ... M
$P_n^m = \left(1 + \frac{\alpha}{\beta}\right)\,\sin\varphi\, P_{n-1}^m \,-\, \frac{\beta}{\alpha} P_{n-2}^m$
$\mathrm{Re}\, Y_{m+1}^m = R_m^{m+1}\, P_{m+1}^m\, C_m$
$\mathrm{Im}\, Y_{m+1}^m = R_m^{m+1}\, P_{m+1}^m\, S_m$
$\alpha = \alpha + 1$
$\beta = \beta + 1$
end do
end do

The atmospheric pressure in the center of each cell, except the south pole, is computed using the values at four corners with using bi-linear interpolation:

$$P = \frac{P_{00} + P_{01} + P_{10} + P_{11}}{4} \tag{10}$$

The cell at the South pole is the disk with radius equal to the space grid. Since all non-zonal harmonics vanish at the pole, and the un-normalized Legendre polynomial is 1, assuming the pressure is linearly changes with latitude, the contribution of the cell at the southern pole is

$$\Delta \mathrm{Re}\, A_0^n = \frac{3}{4\pi\, \bar{R}\, \rho_\oplus\, g_o}\, \frac{1 + k_n'}{2n+1}\, \pi\, \frac{2\, P(-\pi/2, 0) + P(-\pi/2 + \Delta\varphi, 0)}{3}\, R_0^n\, \Delta^2\varphi \tag{11}$$

References

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